利用福里哀级数的(k, φ)平均数逼近连续周期函数

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设$f(x)$是以2π为周期的连续周期函数(记为$f(x)\in C_{2\pi}$)，若对于任意x和h满足条件：

$$|f(x+h)-2f(x)+f(x-h)|\leq M|h|,$$

则称函数$f(x)$属于$H_{\frac{1}{2}}$类，并记为$f(x)\in H_{\frac{1}{2}}$

函数$\varphi(x)$在$[0,1]$区间上定义并为黎曼可积，且

$$\int_0^1\varphi^2(x)\,dx>0,\quad \sum_{k=0}^{n+2}\varphi^2\left(\frac{s}{n+2}\right)>0,\qquad n(=0,1,2,\cdots)$$

此外，置 $A_K^{(n)}=\sum\limits_{s=0}^{n-k+2}\varphi\left(\frac{s}{n+2}\right)\varphi\left(\frac{s+k}{n+2}\right)$，$A_n=A_0^{(n)}$，

$$(k=0,1,2,\cdots;\ n=0,1,2,\cdots);$$

$$\rho_K^{(n)}=\begin{cases}A_K^{(n)}/A_n, & 当k\leq n+1时,\\ 0 & 当k>n+1时,\end{cases}$$

若对于级数$\sum\limits_{k-0}^{\infty}u_K$有且仅仅有

$$\lim_{n\to\infty}\sum_{k=0}^{n}\rho_K^{(n)}u_k=s\,(有限),$$

则称级数$\sum u_k$是(k,φ)可和于s，并记为

$$(k,\varphi)\sum_{k=0}^{\infty}u_k=s,$$

本文考虑下列福里哀级数及其共轭级数的(k,φ)求和法的部分和：

$$L_n(f;x)=\frac{\alpha_0}{2}+\sum_{k=1}^{n}\rho_K^{(n)}(\alpha_k\cos kx+b_k\sin kx)\tag{1}$$

与

$$\widetilde{L}_n(f;x)=\sum_{k=1}^{n}\rho_K^{(n)}(\alpha_k\sin kx-b_k\cos kx)\tag{2}$$

对于H_1^2函数类的逼近度，其中a_k、b_k是函数$f(x)\in C_{2\pi}$的福里哀系数，

若$\varphi(x)\in C$对于区间［0，1］上的任意x和h满足下列条件：

$$\left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\varphi(x+ih)\right|\leq M|h|^p,\quad (0<p\leq r),$$

M为常数，则记$\varphi(x)\in Lip_{M[0,1]}^{(r)}P$，

引理1 若 $\varphi(x)\in Lip_{M[0,1]}^{(r)}P$，　$(P=1,2,3,4)$，
则

$$\triangle^P\rho_K^{(n)}=\sum_{i=0}^{P}(-1)^i\binom{p}{i}\rho_{K+i}^{(n)}=O\left(\frac{1}{n^P}\right),$$

证 当$P=1$，2见［1］．　　当$P=3$时

$$\triangle^3\rho_k^{(n)}=\rho_k^{(n)}-3\rho_{k+1}^{(n)}+3\rho_{k+2}^{(n)}-\rho_{k+3}^{(n)}$$

$$=\frac{1}{A_n}\left\{A_k^{(n)}-3A_{k+1}^{(n)}+3A_{k+2}^{(n)}-A_{k+3}^{(n)}\right\}$$

$$=\frac{1}{A_n}\left\{\sum_{S=0}^{n-k+2}\varphi\left(\frac{S}{n+2}\right)\varphi\left(\frac{S+k}{n+2}\right)-3\sum_{S=0}^{n-k+1}\varphi\left(\frac{S}{n+2}\right)\varphi\left(\frac{S+k+1}{n+2}\right)\right.$$

$$\left.+3\sum_{S=0}^{n-K}\varphi\left(\frac{S}{n+2}\right)\varphi\left(\frac{S+k+2}{n+2}\right)-\sum_{S=0}^{n-k+1}\varphi\left(\frac{S}{n+2}\right)\varphi\left(\frac{S+k+3}{n+2}\right)\right\}$$

$$=\frac{1}{An}\left\{\sum_{S=0}^{n-k-1}\varphi\left(\frac{S}{n+2}\right)\left[\varphi\left(\frac{S+k}{n+2}\right)-3\varphi\left(\frac{S+k+1}{n+2}\right)+3\varphi\left(\frac{S+k+2}{n+2}\right)\right.\right.$$

$$\left.-\varphi\left(\frac{s+k+3}{n+2}\right)\right]+\varphi\left(\frac{n-k}{n+2}\right)\varphi\left(\frac{n}{n+2}\right)+\varphi\left(\frac{n+k+1}{n+2}\right)\varphi\left(\frac{n+1}{n+2}\right)$$

$$+\varphi\left(\frac{n-k+2}{n+2}\right)\varphi\left(\frac{n+2}{n+2}\right)-3\varphi\left(\frac{n-k}{n+2}\right)\varphi\left(\frac{n+1}{n+2}\right)$$

$$\left.-3\varphi\left(\frac{n-k+1}{n+2}\right)\varphi\left(\frac{n+2}{n+2}\right)+3\varphi\left(\frac{n-k}{n+2}\right)\varphi\left(\frac{n+2}{n+2}\right)\right\}$$

显然有

$$\left|\triangle^3\rho_k^{(n)}\right|\leqslant\frac{1}{A_n}\left\{\frac{LM_3}{(n+2)^3}(n-k)+\frac{2LM_2}{(n+2)^2}+\frac{M_1^2}{(n+2)^2}\right\}$$

其中$L=\underset{0\leqslant x\leqslant 1}{Max}|\varphi(x)|$

由于

$$\frac{1}{A_n}=O\left(\frac{1}{n}\right)$$

故

$$\triangle^3\rho_K^{(n)}=O\left(\frac{1}{n}\cdot\frac{1}{(n+2)^2}\right)=O\left(\frac{1}{n^3}\right)$$

当$P=4$时，类似方法可得

$$\triangle^4\rho_K^{(n)}=O\left(\frac{1}{n^4}\right)$$

引理 2　　若　$f(x)\in C_{2\pi}$，则

$$I_1=\int_{\frac{1}{n}}^{X}\sum_{K=3}^{n}\triangle^4\rho_{n-K}^{(n)}\frac{\varphi_x(t)}{(2\sin\frac{t}{2})^4}\cos(n-k+2)tdt$$

$$=O\left(\omega_2\left(\frac{1}{n},f\right)\right)\qquad(4)$$

其中$\varphi_x(t)=f(x+t)-2f(x)+f(x-t)$，$\omega_2(\frac{1}{n},f)$为$f(x)$的光滑模，

证　设$T_n(x)$是$f(x)$的最佳逼近三角多项式，其阶数不高于n，由〔6〕知道

$$E_n(f)=\|f(x)-T_n(x)\|\leqslant C\omega_2\left(\frac{1}{n},f\right),$$

于是有　$\varphi_x(t)=T_n(x+t)-2T_n(x)+T_n(x-t)+O\left(\omega_2(\frac{1}{n},f)\right)$

$$=\psi_n(t)+O\left(\omega_2(\frac{1}{n},f)\right)$$

这里，记　$\psi_n(t)=T_n(x+t)-2T_n(x)+T_n(x-t)$，

且　　$\psi_n(t)=t^2T_n''(x+2\theta t)$，$(|\theta|<1)$，

和　　$|T_n''(x)|\leqslant C_1n^2\omega_2(\frac{1}{n},f)$

于是$|\psi_n(t)|\leqslant C_2n^2t^2\omega_2\left(\frac{1}{n},f\right)$

从而$I_1=\sum\limits_{k=3}^{n}\int_{\frac{1}{n}}^{\pi}\triangle^4\rho_{n-k}^{(n)}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^4}\cos(n-k+2)tdt$

$$=O\left(\sum_{k=o}^{n-3}\left|\triangle^4\rho_k^{(n)}\right|\int_{\frac{1}{n}}^{\pi}\frac{|\psi_n(t)|}{t^4}dt+O\left(\sum_{K=0}^{n-3}\left|\triangle^4\rho_k^{(n)}\right|\int_{\frac{1}{n}}^{\pi}\omega_2(\frac{1}{n},f)\frac{dt}{t^4}\right)\right.$$

$$=O\left(n^3\sum_{k=o}^{n-3}\left|\triangle^4\rho_k^{(n)}\right|\omega_2(\frac{1}{n},f)\right),\qquad(5)$$

由引理1，有　$|\triangle^4\rho_k^{(n)}|\leqslant\frac{C}{n^4}$，

故　$$\sum_{k=o}^{n-3}\left|\triangle^4\rho_K^{(n)}\right|\leqslant\frac{C}{n^4}(n-2)=O\left(\frac{1}{n^3}\right)$$

以之代入(5)式，即得

$$I_1=O\left(\omega_2(\frac{1}{n},f)\right).$$

利用公式 $\dfrac{1}{\left(2\sin\dfrac{t}{2}\right)^2}=\sum\limits_{k=-\infty}^{+\infty}\dfrac{1}{(2k\pi+t)^2}$

和 $\max\limits_{|t|\leq h}|\varphi_x(t)|\leq\sup\limits_{|t|\leq h}\|f(x+t)-2f(x)+f(x-t)\|$
$=O[\omega_2(h,f)]$,

由〔2〕 即可得到：

引理3 若$f(x)\in C_{2\pi}$,则

$$\frac{1}{n+1}\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^2}dt$$

$$=\int_a^{+\infty}\frac{f\left(x+\frac{t}{n}\right)-2f(x)+f\left(x-\frac{t}{n}\right)}{t^2}dt+O\left(\omega_2\left(\frac{1}{n},f\right)\right),$$

这里$a>0$为任意常数，

下面证明本文的结果：

定理1 若$f(x)\in C_2\pi$，$\omega_2\left(\frac{1}{n},f\right)$为$f(x)$的光滑模，则

$$L_n(f;x)-f(x)=\frac{C_n}{\pi}\int_a^{+\infty}\frac{f\left(x+\frac{t}{n}\right)-2f(x)+f\left(x-\frac{t}{n}\right)}{t^2}dt$$
$$+O\left(\omega_2\left(\frac{1}{n},f\right)\right),\tag{6}$$

其中$a>0$为任意常数，$C_n=\frac{n+1}{2}(1-\rho_2^{(n)})$，

证 显然(1)式可表为

$$L_n(f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\sum_{k=0}^{n}\lambda_K^{(n)}D_K(t-x)+\lambda_{n+1}^{(n)}D_n(t-x)\right]dt,\tag{7}$$

其中 $\lambda_K^{(n)}=\rho_K^{(n)}-\rho_{K+1}^{(n)}$, $D_k(t-x)=\dfrac{\sin\left(k+\frac{1}{2}\right)(t-x)}{2\sin\frac{t-x}{2}}$

$$L_n(f;x)-f(x)=\frac{1}{\pi}\int_0^{\pi}[f(x+t)-2f(x)+f(x-t)]K_n(t)dt.\tag{8}$$

积分核 $K_n(t)=\sum\limits_{R=0}^{n}\lambda_K^{(n)}D_K(t)+\lambda_{n+1}^{(n)}D_n(t)$ 并可表为：

$$K_n(t)=I_m\left\{\frac{e^{i(n+\frac{1}{2})t}}{2\sin\frac{t}{2}}\left[\sum_{K=0}^{n}\lambda_{n-k}^{(n)}e^{-ikt}+\lambda_{n+1}^{(n)}\right]\right\},\tag{9}$$

对 $\sum\limits_{K=0}^{n}\lambda_{n-k}^{(n)}e^{-ikt}$进行如下的阿贝尔变换：

$$\sum_{K=m}^{n} u_{n-K}v_k=\sum_{K=m}^{n-1} V_K(u_{n-K}-u_{n-K-1})+V_n u_0-V_{m-1}u_{n-m},$$

这里　$V_K=v_0+v_1+\cdots+v_K$；　$V_{-1}=0$。

注意到　$\sum_{K=0}^{n-1}\left(\lambda_{n-k}^{(n)}-\lambda_{n-k-1}^{(n)}\right)=\lambda_n^{(n)}-\lambda_0^{(n)}$，

所以

$$\sum_{k=0}^{n}\lambda_{n-k}^{(n)}e^{-ikt}=\frac{1}{1-e^{-it}}\triangle\rho_n^{(n)}+\frac{e^{-it}}{(1-e^{-it})^2}\triangle^2\rho_{n-1}^{(n)}$$

$$+\frac{e^{-2it}}{(1-e^{-it})^3}\triangle^3\rho_{n-2}^{(n)}-\frac{e^{-i(n+1)t}}{1-e^{-it}}\triangle\rho_0^{(n)}-\frac{e^{-i(n+1)t}}{(1-e^{-it})^2}\triangle^2\rho_0^{(n)}$$

$$-\frac{e^{-i(n+1)t}}{(1-e^{-it})^3}\triangle^3\rho_0^{(n)}+\sum_{K=3}^{n}\frac{e^{-ikt}}{(1-e^{-it})^3}\triangle^4\rho_{n-k}^{(n)}.\tag{10}$$

把（8）式写为

$$L_n(f;x)-f(x)=\frac{1}{\pi}\left[\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\pi}\right]\varphi_x(t)K_n(t)\,dt。\tag{11}$$

而

$$|K_n(t)|\leqslant\sum_{k=0}^{n}|\lambda_K^{(n)}|\,|D_k(t)|+|\lambda_{n+1}^{(n)}|\,|D_n(t)|$$

$$\leqslant Cn\sum_{k=0}^{n}|\lambda_K^{(n)}|+O(1)$$

由（3）式　出 $\sum_{k=0}^{n}|\lambda_k(n)|=O(1)$，故

$$|k_n(t)|\leqslant Cn,$$

C为与n无关的常数.

因此有

$$\int_0^{\frac{1}{n}}\varphi_x(t)K_n(t)dt=O\left(\omega_2\left(\frac{1}{n},\ f\right)\right)\tag{12}$$

把（10）式代入（9）式中，得

$$K_n(t)=I_m\left\{\frac{e^{i(n+\frac{1}{2})t}}{2\sin\frac{t}{2}}\left[\frac{-ie^{i\frac{t}{2}}}{2\sin\frac{t}{2}}\triangle\rho_n^{(n)}+\frac{1}{(2\sin\frac{t}{2})^2}\triangle^2\rho_{n-1}^{(n)}\right.\right.$$

$$+\frac{-ie^{-i\frac{t}{2}}}{\left(2\sin\frac{t}{2}\right)^3}\triangle\rho^3{}_{n-2}^{(n)}-\frac{-ie^{-i(n+\frac{1}{2})t}}{2\sin\frac{t}{2}}\triangle\rho_0^{(n)}-\frac{e^{-int}}{\left(2\sin\frac{t}{2}\right)^2}\triangle^2\rho_0^{(n)}$$

$$\left.-\frac{-ie^{-i(n-\frac{1}{2})t}}{\left(2\sin\frac{t}{2}\right)^3}\triangle^3\rho_0^{(n)}+\sum_{k=3}^{n}\frac{ie^{-i(k-\frac{3}{2})t}}{\left(2\sin\frac{t}{2}\right)^3}\triangle^4\rho_{n-k}^{(n)}+\rho_{n-1}^{(n)}\right]$$

$$=-\frac{\cos(n+1)t}{\left(2\sin\frac{t}{2}\right)^2}+\frac{\sin\left(n+\frac{1}{2}\right)t}{\left(2\sin\frac{t}{2}\right)^3}\triangle^2\rho_{n-1}^{(n)}-\frac{\cos nt}{\left(2\sin\frac{t}{2}\right)^4}\triangle^3\rho_{n-2}^{(n)}$$

$$+\frac{1}{\left(2\sin\frac{t}{2}\right)^2}\triangle\rho_0^{(n)}-\frac{\sin\frac{t}{2}}{\left(2\sin\frac{t}{2}\right)^3}\triangle^2\rho_0^{(n)}+\frac{\cos t}{\left(2\sin\frac{t}{2}\right)^4}\triangle^3\rho_0^{(n)}$$

$$-\sum_{K=3}^{n}\frac{\cos(n-k+2)t}{\left(2\sin\frac{t}{2}\right)^4}\triangle^4\rho_{n-1}^{(n)}+\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\rho_{n+1}^{(n)}.$$

$$=\frac{1}{2}(1-\rho_2^{(n)})\frac{1}{\left(2\sin\frac{t}{2}\right)^2}-\frac{\cos(n+1)t}{\left(2\sin\frac{t}{2}\right)^2}\triangle\rho_n^{(n)}$$

$$+\frac{\sin\left(n+\frac{1}{2}\right)t}{\left(2\sin\frac{t}{2}\right)^3}\triangle^2\rho_{n-1}^{(n)}-\frac{\cos nt}{\left(2\sin\frac{t}{2}\right)^4}\triangle^3\rho_{n-2}^{(n)}$$

$$+\frac{\cos t}{\left(2\sin\frac{t}{2}\right)^4}\triangle^3\rho_0^{(n)}-\sum_{K=3}^{n}\frac{\cos(n-k+2)t}{\left(2\sin\frac{t}{2}\right)^4}\triangle^4\rho_{n-k}^{(n)}+O(1).$$

由［3］可类似地证明如下两个估计式：

$$\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^2}\cos(n+1)t\,dt=O\left(n\omega_2\left(\frac{1}{n},f\right)\right)\tag{13}$$

及

$$\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^3}\sin\left(n+\frac{1}{2}\right)t\,dt=O\left(n^2\omega_2\left(\frac{1}{n},f\right)\right).\tag{14}$$

把$K_n(t)$代入(11)中，并利用(4)、(12)、(13)、(14)而得

$$L_n(f;x)-f(x)=\frac{1-\rho_2^{(n)}}{2\pi}\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^2}dt-\frac{\triangle^3\rho_{n-2}^{(n)}}{\pi}.$$

$$\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^4}\cos nt\,dt+\frac{\triangle^3\rho_0^{(n)}}{\pi}\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^4}\cos t\,dt+O\left(\omega_2\left(\frac{1}{n},f\right)\right)$$

$$+O\left(n\left|\triangle\rho_n^{(n)}\right|\omega_2\left(\frac{1}{n},f\right)\right)+O\left(n^2\left|\triangle^2\rho_{n-1}^{(n)}\right|\omega_2\left(\frac{1}{n},f\right)\right)$$

$$=\frac{1-\rho_2^{(n)}}{2\pi}\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^2}dt+O\left(\left|\triangle^3\rho_{n-2}^{(n)}\right|\int_{\frac{1}{n}}^{\pi}\frac{|\psi_n(t)|}{t^4}dt\right)$$

$$+O\left(|\triangle^2\rho_0^{(n)}|\int_{\frac{1}{n}}^{\pi}\frac{|\psi_n(t)|}{t^4}dt\right)+O\left(n|\triangle\rho_n^{(n)}|\omega_2(\frac{1}{n},f)\right)$$

$$+O\left(n^2|\triangle^2\rho_{n-1}^{(n)}|\omega_2(\frac{1}{n},f)\right)+O\left(\omega_2\frac{1}{n},f)\right)$$

$$=\frac{1-\rho_2^{(m)}}{2\pi}\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^2}dt+O\left(\omega_2(\frac{1}{n},f)\right).$$

再由引理3即得

$$L_n(f;x)-f(x)=\frac{(n+1)(1-\rho_2^{(n)})}{2\pi}\int_a^{+\infty}\frac{f\left(x+\frac{t}{n}\right)-2f(x)+f\left(x-\frac{t}{n}\right)}{t^2}dt$$

$$+O\left(\omega_2(\frac{1}{n},f)\right)$$

$$=\frac{C_n}{\pi}\int_a^{+\infty}\frac{f\left(x+\frac{t}{n}\right)-2f(x)+f\left(x-\frac{t}{n}\right)}{t^2}dt+O\left(\omega_2(\frac{1}{n}.f)\right).$$

其中$\alpha>0$为任意常数，$C_n=\frac{(1+n)(1-\rho^{(n)})}{2}$

推论1　当$x\in(0,1)$时，令$\varphi(x)=1$，而$\varphi(0)=\varphi(1)=0$，显然，(K,φ)求和法就是$(C,1)$求和法，而

$$C_n=\frac{(1+n)(1-\rho^{(n)})}{2}=\frac{1+n}{2}(1-A_2^{(n)}/A_n)$$

$$=\frac{1+n}{2}\left(1-\sum_{S=0}^{n}\varphi\left(\frac{s}{n+2}\right)\varphi\left(\frac{s+2}{n+2}\right)\Big/\sum_{S=0}^{n+2}\varphi^2\left(\frac{s}{n+2}\right)\right)$$

$$=\frac{1+n}{2}\left(1-\frac{n-1}{n+1}\right)=1.$$

这时定理1成为

$$L_n(f;x)-f(x)=\frac{1}{\pi}\int_a^{+\infty}\frac{f\left(x+\frac{t}{n}\right)-2f(x)+f\left(x-\frac{t}{n}\right)}{t^2}dt+O\left(\omega_2(\frac{1}{n},f)\right),$$

这就是叶菲莫夫〔4〕定理1的结果.

定理2　若$f(x)\in H_2^1$，则

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{1}{\pi}\sum_{k=0}^{n+1}\triangle^2\rho_K\int_0^{\frac{a_1}{K+1}}\psi(t)\frac{\sin(k+1)t}{t_2}dt+O\left(\frac{1}{n}\right).$$

其中　$\psi(t)=f(x+t)-f(x-t)$,

$$\widetilde{f}(x)=-\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{1}{2}\mathrm{ctg}\frac{t}{2}dt,$$

$a_1>0$　为方程$\int_0^{\pi}\frac{\sin t}{t}dt=\frac{\pi}{2}$最小的根.

证　由　$\widetilde{L}_n(f;x = \sum_{k=0}^{n}\rho_k^{(n)}(a_k\sin kt-\rho_k\cos kt)$，

则　$$\widetilde{L}_n(f;x)=-\frac{1}{\pi}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\rho_k^{(n)}\sin kt dt$$

$$=-\frac{1}{\pi}\int\psi(t)\left[\sum_{K=1}^{n}\lambda_k^{(n)}\widetilde{D}_K(t)+\lambda_{n+1}^{(n)}\widetilde{D}_n(t)\right]dt.$$

其中　$\widetilde{D}_K(t)=\dfrac{\cos\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$.　　于是

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=-\frac{1}{\pi}\int_0^{\pi}\varphi(t)\left[\sum_{K=1}^{n}\lambda_K^{(n)}\widetilde{D}_K(t)+\lambda_{n+1}^{(n)}\widetilde{D}_n(t)-\frac{1}{2}\operatorname{ctg}\frac{t}{2}\right]dt$$

$$=\frac{1}{\pi}\int_0^{\pi}\psi(t)\sum_{K=0}^{n}\lambda_{n-K}^{(n)}\frac{\cos(n-k+\frac{1}{2})t}{2\sin\frac{t}{2}}dt$$

$$+\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}dt$$

由于 $\lambda_{n+1}^{(n)}=O\left(\frac{1}{n}\right)$及$\psi(t)=O\left(t\log\frac{1}{t}\right)$. 所以

$$\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}dt=O\left(\frac{1}{n}\right),$$

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\varphi(t)\widetilde{K}_n(t)dt+O\left(\frac{1}{n}\right), \tag{15}$$

其中$\widetilde{K}_n(t)=\sum_{K=0}^{n}\lambda_{n-K}^{(n)}\dfrac{\cos(n-k+\frac{1}{2})t}{2\sin\frac{t}{2}}$，并表为

$$\widetilde{K}_n(t)=\operatorname{Re}\left\{\frac{e^{i(n+\frac{1}{2})t}}{2\sin\frac{t}{2}}\sum_{K=0}^{n}\lambda_{n-K}^{(n)}e^{-iKt}\right\}.$$

利用阿贝尔变换得

$$\widetilde{K}_n(t)=\operatorname{Re}\left\{\frac{e^{i(n+\frac{1}{2})t}}{2\sin\frac{t}{2}}\left[\sum_{k=0}^{n}(\lambda_{n-K}^{(n)}-\lambda_{n-K-1}^{(n)})\frac{1-e^{-i(K+1)t}}{1-e^{-it}}\right.\right.$$

$$\left.\left.+\frac{1-e^{-i(n+1)t}}{1-e^{-it}}\lambda_0^{(n)}\right]\right\}$$

$$=\frac{\sin(n+1)t}{2\sin\frac{t}{2}}\lambda_n^{(n)}+\sum_{k=0}^{n-1}\triangle^2\rho_{n-k+1}^{(n)}\frac{\sin(n-k)t}{\left(2\sin\frac{t}{2}\right)^2}$$

把$\widetilde{K}_n(t)$代入(15)中得

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{\lambda_n^{(n)}}{\pi}\int_0^\pi\psi(t)\frac{\sin(n+1)t}{\left(2\sin\frac{t}{2}\right)^2}dt$$

$$+\frac{1}{\pi}\sum_{k=0}^{n-1}\triangle^2\rho_k^{(n)}\int_0^\pi\psi(t)\frac{\sin(k+1)t}{\left(2\sin\frac{t}{2}\right)^2}dt+O\left(\frac{1}{n}\right)$$

$$=\frac{\lambda_n^{(n)}}{\pi}\int_0^\pi\psi(t)\frac{\sin(n+1)t}{\left(2\sin\frac{t}{2}\right)^2}dt+\frac{1}{\pi}\sum_{k=0}^{n-1}\triangle^2\rho_k^{(n)}\int_0^\pi\psi(t)$$

$$\frac{\sin(k+1)t}{\left(2\sin\frac{t}{2}\right)^2}dt+O\left(\frac{1}{n}\right).$$

由〔4〕可知，当 $f(x)\in H_{\frac{1}{2}}$则有

$$\int_{\frac{a_1}{k}}^{\pi}\psi(t)\frac{\sin kt}{t^2}dt=O(1).$$

利用这个结果与引理1，即有

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{\lambda_n^{(n)}}{\pi}\left[\int_0^{\frac{a_1}{n+1}}+\int_{\frac{a_1}{n+1}}^{\pi}\right]\psi(t)\frac{\sin(n+1)t}{t^2}dt$$

$$\frac{1}{\pi}\sum_{K=0}^{n-1}\triangle^2\rho_K^{(n)}\left[\int_0^{\frac{a_1}{K+1}}+\int_{\frac{a_1}{K+1}}^{\pi}\right]\psi(t)\frac{\sin(k+1)t}{t^2}dt+O\left(\frac{1}{n}\right)$$

$$=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\frac{a_1}{n+1}}\psi(t)\frac{\sin(n+1)t}{t^2}\rho t+\frac{1}{\pi}\sum_{K=0}^{n}\triangle^2\rho_k^{(n)}\int_0^{\frac{a_1}{K+1}}\psi(t)\frac{\sin(k+1)t}{t^2}dt$$

$$+O\left(\frac{1}{n}\right)\qquad(16)$$

而

$$I=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\frac{a_1}{n+1}}\psi(t)\frac{\sin(n+1)t}{t^2}dt$$

$$=\frac{\lambda_{n+1}^{(n)}}{\pi}\left[\int_0^{\frac{a_1}{n+2}}\psi(t)\frac{\sin(n+2)t}{t^2}dt+\int_{\frac{a_1}{n+2}}^{\frac{a_1}{n+1}}\psi(t)\frac{\sin(n+1)t}{t^2}dt\right.$$

$$\left.+\int_0^{\frac{a_1}{n+1}}\psi(t)\frac{\sin(n+1)t-\sin(n+2)t}{t^2}dt\right]$$

$$=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\frac{a_1}{n+2}}\psi(t)\frac{\sin(n+2)t}{t^2}dt+I_1+I_2\qquad(17)$$

由于 $\psi(t)=O(1)$,

$$I_1=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_{\frac{n_1}{n+2}}^{\frac{a_1}{n+1}}\psi(t)\frac{\sin(n+1)t}{t^2}=O\left(\frac{1}{n}\int_{\frac{a_1}{n+2}}^{\frac{a_1}{n+1}}\frac{dt}{t^2}\right)$$

$$=O\left(\frac{1}{n}\right).$$

$$I_2=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\frac{a_1}{n+2}}\psi(t)\frac{\sin(n+1)t-\sin(n+2)t}{t^2}dt$$

$$=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\frac{a_1}{n+2}}\psi(t)\frac{2\cos\left(n+\frac{3}{2}\right)t\sin t}{t^2}dt$$

$$=O\left(\frac{1}{n}\int_0^{\frac{a_1}{n+2}}\log\frac{1}{t}dt\right)$$

$$=O\left(\frac{1}{n}t\log\frac{1}{t}\Big|_0^{\frac{a_1}{n+2}}-\int_0^{\frac{a_1}{n+2}}dt\right)$$

$$=O\left(\frac{1}{n}\right)$$

把I_1、I_2的估计式代入(17)中，得

$$I=\frac{\lambda_{n+1}^{(n)}}{\pi}\int_0^{\frac{a_1}{n+2}}\psi(t)\frac{\sin(n+2)t}{t^2}dt+O\left(\frac{1}{n}\right)$$

把I代入(16)中，并由于$\lambda_{n+2}^{(n)}=0$即得

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{1}{\pi}\sum_{k=0}^{n+1}\triangle^2\rho_k^{(n)}\int_0^{\frac{a_1}{k+1}}\psi(t)\frac{\sin(k+1)t}{t^2}dt+O\left(\frac{1}{n}\right)$$

推论 2 令 $\varphi(x)=\begin{cases}1 & 当0<x<1时,\\ 0 & 当x=0,\ x=1时.\end{cases}$

则$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{1}{(1+n)\pi}\int_0^{\frac{1}{n+1}}\psi(t)\frac{\sin(n+1)t}{t^2}dt+O\left(\frac{1}{n}\right)$，这就是叶菲莫夫

[4] 的定理2．事实上，这时对于所有$k\leqslant n-1$皆成立

$$\triangle^2\rho_k^{(n)}=\rho_k^{(n)}-2\rho_{k+1}^{(n)}+\rho_{k+2}^{(n)}$$

$$=\frac{1}{A_n}\left\{A_k^{(n)}-2A_{k+1}^{(n)}+A_{k+2}^{(n)}\right\}$$

$$=\left\{\sum_{S=0}^{n-k+2}\varphi\left(\frac{s}{n+2}\right)\varphi\left(\frac{s+k}{n+2}\right)-2\sum_{S=0}^{n-k+1}\varphi\left(\frac{s}{n+2}\right)\varphi\left(\frac{s+k+1}{n+2}\right)\right.$$

$$+\sum_{S=0}^{n-k}\varphi\left(\frac{s}{n+2}\right)\varphi\left(\frac{s+k+2}{n+2}\right)\Big\}\Big/\sum_{S=0}^{n+2}\varphi^2\left(\frac{s}{n+2}\right)$$

$$=\frac{1}{n+1}\Big\{(n-k+1)-2(n-k)+(n-k+1)\Big\}=0.$$

当$k=n$时，　$\triangle^2\rho_n^{(n)}=\rho_n^{(n)}-2\rho_{n+1}^{(n)}=\frac{1}{1+n}.$

当$k=n+1$时，　$\triangle^2\rho_{n+1}^{(n)}=\rho_{n+1}^{(n)}=0.$

这时定理2成为

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{1}{(1+n)\pi}\int_0^{\frac{a}{n+1}}\varphi(t)\frac{\sin(n+1)t}{t^2}dt+O\left(\frac{1}{n}\right).$$

定理3　以下估计式成立

$$E_{\widetilde{L}_n}(H_{\frac{1}{2}})=\sup_{f\in H_{\frac{1}{2}}}\left\|\widetilde{L}_n(f;x)-\widetilde{f}(x)\right\|$$

$$=\frac{1}{2\ln(\sqrt{2}+1)}\sum_{k=0}^{n+1}|\triangle_2\rho_k^{(n)}|\ln(1+K)+O\left(\frac{1}{n}\right)$$

证　由［5］有

$$|f(x+t)-f(x-t)|\leqslant\frac{1}{2\ln(\sqrt{2}+1)}2t\ln\frac{1}{t}+O(t).$$

令　$g(t)=\int_0^t\frac{\sin u}{u}du$，　则

$$g(0)=0,\quad g(a_1)=\frac{\pi}{2}.$$

由定理2的结果有

$$\left|\widetilde{L}_n(f;x)-\widetilde{f}(x)\right|\leqslant\frac{1}{\pi\ln(\sqrt{2}+1)}\sum_{k=0}^{n+1}|\triangle^2\rho_k^{(n)}|\int_0^{\frac{a_1}{k+1}}\ln\frac{1}{2t}\frac{\sin(k+1)t}{t}dt+O\left(\frac{1}{n}\right)$$

$$=\frac{1}{\pi\ln(\sqrt{2}+1)}\sum_{k=0}^{n+1}|\triangle^2\rho_k^{(n)}|\int_0^{a_1}\ln\frac{k+1}{2t}\frac{\sin t}{t}dt+O\left(\frac{1}{n}\right)$$

$$=\frac{1}{\pi\ln(\sqrt{2}+1)}\sum_{k=0}^{n+1}|\triangle^2\rho_k^{(n)}|\left[g(t)\ln\frac{k+1}{2t}\Big|_0^{a_1}dt\right]+O\left(\frac{1}{n}\right)$$

$$=\frac{1}{2\pi\ln(\sqrt{2}+1)}\sum_{k=0}^{n+1}|\triangle^2\rho_k^{(n)}|\ln(k+1)+O\left(\frac{1}{n}\right).$$

推论3　当$\varphi(x)=\begin{cases}1, & \text{当 } 0<x<1\\ 0, & x=0,\ x=1\end{cases}$时，

$$\widetilde{L}_n(f;x)-\widetilde{f}(x)=\frac{1}{2\pi\ln(\sqrt{2}+1}\frac{\ln(n+1)}{n+1}+0\left(\frac{1}{n}\right).$$

即为叶菲莫夫〔4〕定理3的结果。

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