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样条有限点法求解任意四边形薄板大挠度问题

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摘 要 本文利用坐标变换下的样条有限点法，研究了任意四边形薄板的大挠度计算问题，给出了计算实例，验证了方法的正确性和较好的通用性。

关键词 任意四边形薄板；大挠度；样条有限点法

中图分类号 O647.2

引言

对矩形薄板的大挠度问题有各种各样的求解方法，这些方法大都受到"矩形"这个几何形状的限制，而对任意四边形薄板的计算，除有限元法以外，还没有其它较通用的解决方法，只有少数方法解决了某些特殊的问题，如文献[1]采用摄动法求解了四边固定梯形板的大挠度问题。同时，有限元法有输入数据多，计算复杂等缺点。

样条有限点法[3]采用梁函数级数作为某一方向上的形函数集，而在垂直于此方向上采用B样条插值函数，只要对靠近边界的基样条给予适当修正，样条有限点法就适应于不同类型边界的情况[4]。

本文采用坐标变换下的样条有限点法，将任意四边形变换成正方形，在正方形板上假设三个方向的位移函数，挠度函数采用四次变换，水平位移采用二次变换。本文最后给出了算例，并和文[1]的结果进行了比较，表明本文方法求解任意四边形薄板大挠度问题是正确而有效的。

1 公式推导

设任意四边形薄板受分布荷载 q（x，y）作用，系统的总势能为：

$$
\begin{aligned}
\Pi = {} & \frac{1}{2}\iint_R \{X\}^T[D]\{X\}\,dxdy + \frac{Eh}{2(1-\mu^2)}\iint_R \Big\{\Big(\frac{\partial u}{\partial x}\Big)^2 + \frac{\partial u}{\partial x}\Big(\frac{\partial w}{\partial x}\Big)^2 \\
& + 2\mu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x}\Big(\frac{\partial w}{\partial y}\Big)^2 + \mu\frac{\partial v}{\partial y}\Big(\frac{\partial w}{\partial x}\Big)^2 + \frac{\partial v}{\partial y}\Big(\frac{\partial W}{\partial y}\Big)^2 \\
& + \Big(\frac{\partial v}{\partial y}\Big)^2 + \frac{1-\mu}{2}\Big[\Big(\frac{\partial u}{\partial y}\Big)^2 + 2\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
& + 2\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \Big(\frac{\partial v}{\partial x}\Big)^2\Big]\Big\}dxdy - \iint_R Wq\,dxdy
\end{aligned}
\tag{1}
$$

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采用如下位移函数及坐标变换：

$$\left.\begin{aligned} W(x,y) &= J^4 W^*(\xi,\eta) = J^4 \sum_{m=1}^{r} \{c\}_m [\Phi] Z_m \\ U(x,y) &= J^2 U^*(\xi,\eta) = J^2 \sum_{m=1}^{r} \{a\}_m [\Phi] U_m \\ V(x,y) &= J^2 V^*(\xi,\eta) = J^2 \sum_{m=1}^{r} \{b\}_m [\Phi] V_m \end{aligned}\right\} \tag{2}$$

其中 $\{c\}_m[\Phi]$，$\{a\}_m[\Phi]$，$\{b\}_m[\Phi]$ 均为三次样条基函数的线性组合。则有：

$$\frac{\partial w}{\partial x} = 4J^2\left(\frac{\partial y}{\partial \eta}\frac{\partial J}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial J}{\partial \eta}\right)W^* + J^3\left(\frac{\partial y}{\partial \eta}\frac{\partial w^*}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial w^*}{\partial \eta}\right) \tag{3}$$

同理可求出$\frac{\partial w}{\partial y}$、$\frac{\partial^2 w}{\partial x^2}$、$\frac{\partial^2 w}{\partial y^2}$、$\frac{\partial^2 w}{\partial x \partial y}$以及$\frac{\partial u}{\partial x}$、$\frac{\partial u}{\partial y}$、$\frac{\partial v}{\partial x}$、$\frac{\partial v}{\partial y}$，将上述结果代入(1)即得到坐标变换下的总势能表达式：

$$\begin{aligned} \Pi = {} & \frac{1}{2}\{C\}^T[G]\{C\} + \frac{Eh}{2(1-\mu^2)}\Big[\{a\}^T[G_1]\{a\} + \{c\}^T[G_2]\{c\} \\ & + \{c\}^T[G_3]\{c\} + \{a\}^T[G_4]\{b\} + \{b\}^T[G_5]\{b\}\Big] - \{c\}^T\{f\} \end{aligned} \tag{4}$$

或写成：

$$\begin{aligned} \Pi^* = {} & \frac{1}{2}\{C\}^T[G]\{C\} + \frac{Eh}{2(1-\mu^2)}\Big[\{a\}^T[G_1]\{a\} + \{a\}^T[G_2^*]\{c\} \\ & + \{b\}^T[G_3^*]\{c\} + \{a\}^T[G_4]\{b\} + \{b\}^T[G_5]\{b\}\Big] - \{c\}^T\{f\} \end{aligned} \tag{5}$$

(4)、(5)式是等效的。

由广义势能原理 $\frac{\partial \Pi}{\partial\{c\}} = 0$，$\frac{\partial \Pi^*}{\partial\{a\}} = 0$，$\frac{\partial \Pi^*}{\partial\{b\}} = 0$，得：

$$\left[[G] + \frac{Eh}{1-\mu^2}([G_2] + [G_3])\right]\{c\} = \{f\} \tag{6}$$

$$[G_1]\{a\} + \frac{1}{2}[G_2^*]\{c\} + \frac{1}{2}[G_4]\{b\} = \{0\} \tag{7}$$

$$[G_5]\{b\} + \frac{1}{2}[G_3^*]\{c\} + \frac{1}{2}[G_4^*]\{a\} = \{0\} \tag{8}$$

这就是坐标变换下的任意四边形薄板大挠度问题的三个样条有限点公式。其中刚度阵的具体形式如下：

$$[G] = \int_0^1\int_0^1 [B]^T[A]^T[D][A][B]J\,d\xi\,d\eta \tag{9}$$

$$[A] = \begin{bmatrix} -a_{\xi\xi} & -a_{\eta\eta} & -a_{\xi\eta} & -a_{\xi} & -a_{\eta} & -a_{o} \\ -b_{\xi\xi} & -b_{\eta\eta} & -b_{\xi\eta} & -b_{\xi} & -b_{\eta} & -b_{o} \\ -2c_{\xi\xi} & -2c_{\eta\eta} & -2c_{\xi\eta} & -2c_{\xi} & -2c_{\eta} & -2c_{o} \end{bmatrix} \tag{9a}$$

$$[B] = \left[[B]_1\,[B]_2\cdots\cdots[B]_r\right] \tag{9b}$$

$$[B]_m = \left[[\Phi'']^T Z_m \quad [\Phi]^T Z''_m \quad [\Phi']^T Z'_m \quad [\Phi']^T Z_m \quad [\Phi]^T Z'_m \quad [\Phi]^T Z_m\right]^T \tag{9c}$$

$$[G_1] = \int_0^1\int_0^1 [B_1]^T[A_1]^T[D_1][A_1][B_1]J\,d\xi\,d\eta \tag{10}$$

$$[A_1] = \begin{bmatrix} \frac{\partial y}{\partial \eta}J & -\frac{\partial y}{\partial \xi}J & 2\left(\frac{\partial J}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial J}{\partial \eta}\frac{\partial y}{\partial \xi}\right) \\ -\frac{\partial x}{\partial \eta}J & \frac{\partial x}{\partial \xi}J & 2\left(\frac{\partial J}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial J}{\partial \xi}\frac{\partial x}{\partial \eta}\right) \end{bmatrix} \tag{10a}$$

$$[B_1] = \left[[B]_1\,[B]_2\cdots\cdots[B]_r\right] \tag{10b}$$

$$[B_1]_m = \left[[\Phi']^T U_m \quad [\Phi]^T U'_m \quad [\Phi]^T U_m\right]^T \tag{10c}$$

$$[G_2]=\int_0\int_0[B_2^*]^T[A_2^*]^T[D_2][A_2][B_2]Jd\xi d\eta \tag{11}$$

$$[A_2]=\begin{bmatrix} J^3\frac{\partial y}{\partial\eta} & -J^3\frac{\partial y}{\partial\xi} & 4J^2(\frac{\partial y}{\partial\eta}\frac{\partial z}{\partial\xi}-\frac{\partial y}{\partial\xi}\frac{\partial z}{\partial\eta}) \\ -J^3\frac{\partial x}{\partial\eta} & J^3\frac{\partial x}{\partial\xi} & 4J^2(\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial z}{\partial\xi}) \\ -J^3\frac{\partial x}{\partial\eta} & J^3\frac{\partial x}{\partial\xi} & 4J^2(\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial z}{\partial\xi}) \end{bmatrix} \tag{11a}$$

$$[B_2]=\left[[B_2]_1[B_2]_2\cdots\cdots[B_2]_r\right] \tag{11b}$$

$$[B_2]_m=\left[[\Phi']^TZ_m \quad [\Phi]^TZ'_m \quad [\Phi]^TZ_m\right]^T \tag{11c}$$

$$[A_2^*]=\begin{bmatrix} A_{211}^* & A_{212}^* & \cdots\cdots A_{219}^* \\ A_{221}^* & A_{222}^* & \cdots\cdots A_{229}^* \\ A_{231}^* & A_{232}^* & \cdots\cdots A_{239}^* \end{bmatrix} \tag{11d}$$

$$A_{211}^*=J^4(\frac{\partial y}{\partial\eta})^2,A_{212}^*=-J^4\frac{\partial y}{\partial\eta}\frac{\partial y}{\partial\xi}\cdots\cdots$$

$$[B_2^*]=\left[[B_2^*]_1 \quad [B_2^*]_2 \quad \cdots\cdots[B_2^*]_r\right] \tag{11e}$$

$$[B_2^*]_m=\sum_{i=1}^{r}\Big[([\Phi']u_i\{a\}_i[\Phi'])^TZ_m \quad ([\Phi']u_i\{a\}_i[\Phi])^TZ'_m$$

$$\cdots\cdots([\Phi]u_i\{a\}_i[\Phi])^TZ_m\Big]^T \tag{11f}$$

$$[G_2^*]=\int_0\int_0[B_2^{**}]^T[A_2^*]^T[D_2][A_2][B_2]Jd\xi d\eta \tag{12}$$

$$[B_2^{**}]=\left[[B_2^{**}]_1[B_2^{**}]_2\cdots\cdots[B_2^{**}]_r\right] \tag{12a}$$

$[B_2^{**}]_m$ 同 $[B_2^*]_m$ 只是将其中的 Z_m 变成 U_m，U_i 变成 Z_i，$\{a\}_i$ 变成 $\{c\}_i$。

$$[G_3]=\int_0\int_0[B_3^*]^T[A_3^*]^T[D_2][A_3][B_2]Jd\xi d\eta \tag{13}$$

$$[A_3]=\begin{bmatrix} -J^3\frac{\partial x}{\partial\eta} & J^3\frac{\partial x}{\partial\xi} & 4J^2(\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}-\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}) \\ J^3\frac{\partial y}{\partial\eta} & -J^3\frac{\partial y}{\partial\xi} & 4J^2(\frac{\partial y}{\partial\eta}\frac{\partial z}{\partial\xi}-\frac{\partial y}{\partial\xi}\frac{\partial z}{\partial\eta}) \\ -J^3\frac{\partial x}{\partial\eta} & J^3\frac{\partial x}{\partial\xi} & 4J^2(\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}-\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}) \end{bmatrix} \tag{13a}$$

$$[A_3^*]=\begin{bmatrix} A_{311}^* & A_{312}^* & \cdots\cdots A_{319}^* \\ A_{321}^* & A_{322}^* & \cdots\cdots A_{329}^* \\ A_{331}^* & A_{332}^* & \cdots\cdots A_{339}^* \end{bmatrix} \tag{13b}$$

$$A_{311}^*=J^4(\frac{\partial x}{\partial\eta})^2,A_{312}^*=-J^4\frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta},\cdots\cdots$$

$$[B_3^*]=\left[[B_3^*]_1 \quad [B_3^*]_2 \quad \cdots\cdots[B_3^*]_r\right] \tag{13c}$$

$$[B_3^*]_m=\sum_{i=1}^{r}\Big[([\Phi']v_i\{b\}_i[\Phi'])^TZ'_m \quad ([\Phi']v_i\{b\}_i[\Phi])'Z'_m$$

$$\cdots\cdots([\Phi]v_i\{b\}_i[\Phi])^TZ_m\Big]^T \tag{13d}$$

$$[G_3^*]=\int_0\int_0[B_3^{**}]^T[A_3^*]^T[D_2][A_3][B_2]Jd\xi d\eta \tag{14}$$

$$[B_3^{**}]=\left[[B_3^{**}]_1[B_3^{**}]_2\cdots\cdots[B_3^{**}]_r\right] \tag{14a}$$

$$[B_3^{**}]_m = \sum_{i=1}^{r}\Big[\,([\Phi']Z_i\{c\}_i[\Phi'])^T V_m \quad ([\Phi]Z_i'\{C\}_i[\Phi'])^T V_m$$

$$\cdots\cdots([\Phi]Z_i\{c\}_i[\Phi])^T V_m\Big]^T \tag{14b}$$

$$[G_4] = \int_0\int_0[B_1]^T[A_1]^T[D_3][A_5][B_5]J\,d\xi d\eta \tag{15}$$

$$[B_5] = \big[\,[B_5]_1\,[B_5]_2\cdots\cdots[B_5]_r\big] \tag{15a}$$

$$[B_5]_m = \big[\,[\Phi']^T V_m\,[\Phi]^T V_m'\,[\Phi]^T V_m\big]' \tag{15b}$$

$$[A_5] = \begin{pmatrix} -\frac{\partial x}{\partial\eta}/J & \frac{\partial x}{\partial\xi}/J & 2\left(\frac{\partial x}{\partial\xi}\frac{\partial}{\partial\eta} - \frac{\partial x}{\partial\eta}\frac{\partial}{\partial\xi}\right) \\ \frac{\partial y}{\partial\eta}/J & -\frac{\partial y}{\partial\xi}/J & 2\left(\frac{\partial y}{\partial\eta}\frac{\partial}{\partial\xi} - \frac{\partial y}{\partial\xi}\frac{\partial}{\partial\eta}\right) \end{pmatrix} \tag{15c}$$

$$[G_4^{*}] = \int_0\int_0[B_5]^T[A_5]^T[D_3][A_1][B_1]J\,d\xi d\eta \tag{16}$$

$$[G_5] = \int_0\int_0[B_5]^T[A_5]^T[D_1][A_5][B_5]J\,d\xi d\eta \tag{17}$$

外荷载 $\{f\} = \int_0\int_0 q[N]^T J^5 d\xi d\eta$ (18)

以上各式中［J］为 Jacobi 矩阵，J＝det［J］，

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} \tag{19}$$

［D］为板的弯曲刚度；

$$[D_1] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-\mu}{2} \end{bmatrix},[D_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix},[D_3] = \begin{bmatrix} 2\mu & 0 \\ 0 & 1-\mu \end{bmatrix} \tag{20}$$

$$[N] = \big[\,[\Phi]Z_1 \quad [\Phi]Z_2 \quad \cdots\cdots \quad [\Phi]Z_r\big] \tag{21}$$

$$\{X\} = \left[-\frac{\partial^2 w}{\partial x^2} \quad -\frac{\partial^2 w}{\partial y^2} \quad -\frac{\partial^2 w}{\partial x\partial y}\right]^T = [A][B]\{C\} \tag{22}$$

2　试函数的选择及方程的求解

2.1 试函数的选择

Z_m 的选择取梁的振型函数，例如，若板对边简支，则取 $Z_m = \sin\frac{m\pi}{b}y$，而 U_m 与 V_m 的选择往往取决于荷载分布情况以及板的几何形状，如对边简支可取 $U_m = \sin\frac{m\pi}{b}y$，$V_m = \cos\frac{m\pi}{b}y$，若对边固定，荷载又对称，可取 $U_m = \sin\frac{m\pi}{b}y$，$V_m = \cos\frac{(m+1)\pi}{b}y$。

2.2 方程的求解

方程（6）（7）（8）中的［G_2］［G_2^*］等是随位移改变的，因此本文使用迭代的方法，逐次逼近精确解，具体实施如下：

（a）不考虑非线性项的影响，（7）（8）式变为：

$$[G_1]\{a\}+\frac{1}{2}[G_4]\{b\}=\{0\} \qquad [G_5]\{b\}+\frac{1}{2}[G_4]\{a\}=\{0\}$$

由此解出 $\{a\}=\{b\}=\{0\}$;

(b)　$\{a\}$ $\{b\}$ 隐含在 $[G_2]$ $[G_3]$ 中，将 $\{a\}$ $\{b\}$ 值代入(6)式即可求出 $\{c\}$;

(c)　$\{a\}$ $\{c\}$ 值代入(8)式求得;

$$\{b\}=-\frac{1}{2}[G_5]^{-1}[G_4^{\ *}]\{a\}-\frac{1}{2}[G_5]^{-1}[G_3^{\ *}]\{c\} \tag{23}$$

(d)　将(23)式代入(7)式得:

$$\left([G_1]-\frac{1}{4}[G_4][G_5]^{-1}[G_4^{\ *}]\right)\{a\}=\frac{1}{2}\left(\frac{1}{2}[G_4][G_5]^{-1}[G_3^{\ *}]-[G_2^{\ *}]\right)\{c\} \tag{24}$$

由此解出 $\{a\}$。

逐次循环上述(b)(c)(d)过程，直到 $\{c\}$ 满足要求的精度为止。

3　计算实例

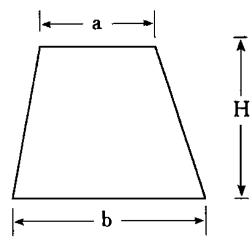

图1　梯形薄板

设有图1所示的梯形薄板，受均布荷载 q 作用。图2给出了四种边界条件下五种不同几何尺寸时中心挠度的计算结果，其中(a)为四边简支，(b)为四边固定，(c)为对边简支对边固定，(d)为三边固定一边自由，几何尺寸为①H＝b＝a，②H＝a，b＝1.5a，③H＝a，b＝2a，④H＝2a，b＝2a，⑤H＝2a，b＝3a，图3给出了四边固定梯形板在均布荷载作用下的中心挠度计算结果，并与文献[1]作了比较，图中Ⅰ为本文解，Ⅱ为文献[1]解。几何尺寸为 b＝2a，H＝a。

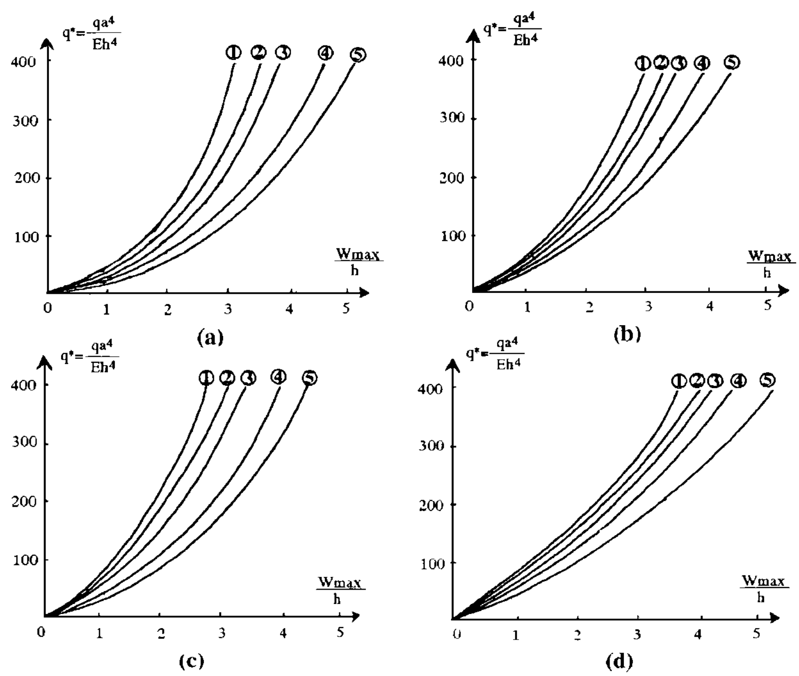

图2　均布荷载作用下梯形薄板的中心挠度

4 结果分析及结论

从上面的图形分析可知，曲线无交叉，当梯形板的几何尺寸增大时，中心挠度也随之增大，特别是当高度增大时，挠度增大更为明显，另外从图3可以看出，本文的计算结果与文献[1]的结果是基本吻合的。

因此本文采用坐标变换的方法计算任意四边形薄板的大挠度问题，计算过程简单，方法正确合理，边界条件易于处理，具有较强的通用性。

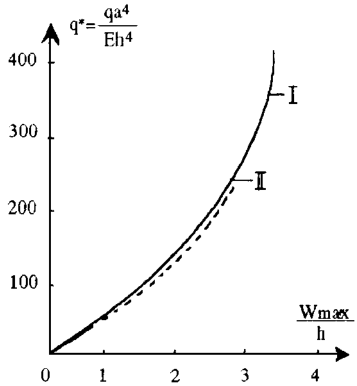

图3　四边固定梯形板中心挠度计算结果比较

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Spline Finite Point Method for Heavy Deflection Problems of Quadrilateral Plates

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Abstract　This paper studied the heavy deflection problems of quadrilateral plates by the spline finite point method under coordinate transform, provided the calculating examples to prove the validity and the good generalpurpose of the method.

Keywords　quadrilateral plates; heavy deflection; spline finite point method.