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On the Divergence Expression and the Boundary Value of Modified Magnetic Vector Potential

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Abstract: In this paper, the divergence expression of modified magnetic vector potential $\vec{A}^*$ in a homogeneous medium is obtained using the time harmonic Maxwell's equations. A boundary value problem of $\vec{A}^*$ is established, and the uniqueness of the solution of the problem is proved. Finally, the validity of the boundary value problem is verified by a time-harmonic field with an analytical solution.

Key words: modified magnetic vector potential; gauge; boundary value problem; time harmonic electromagnetic field
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Introduction

In recent years, several numerical methods have been proposed for calculating the eddy current fields, and these methods usually focus on the electromagnetic potentials which can be used to establish the boundary value problems[1,2]. The widely-used electromagnetic potentials are magnetic vector potential $\vec{A}$, electric scalar potential φ, modified magnetic vector potential $\vec{A}^*$, electric vector potential $\vec{T}$, and magnetic scalar potential Ω. Among these electromagnetic potentials, the definition of modified magnetic vector potential $\vec{A}^*$ [3] is of great interest, because $\vec{A}^*$ includes both magnetic vector potential $\vec{A}$ and electric scalar potential φ, reducing the number of variables that need to be solved. In addition, some boundary value problems become easy to be solved when using the modified magnetic vector $\vec{A}^*$. However, it is noticed that the Coulomb gauge $\nabla\cdot\vec{A}^*=0$ is not always satisfied for the time harmonic electromagnetic fields. Particularly, $\nabla\cdot\vec{A}^*$ will not be zero when the source currents exist. This fact motivates us to reconsider the governing equations of $\vec{A}^*$, boundary conditions of $\vec{A}^*$, and the uniqueness of the solution of $\vec{A}^*$.

The remainder of this paper is as follows. In Sec.1 we give the divergence expression of modified magnetic vector potential $\vec{A}^*$. In Sec.2 we establish the boundary value problem of $\vec{A}^*$. In Sec.3 the uniqueness of the solution of $\vec{A}^*$ is proved. In Sec.4 we give an analytical solution of $\vec{A}^*$ for the Sommerfed's half-space problem. Conclusions are stated in Sec.5.

1 Divergence Expression of $\vec{A}^*$

Consider a time harmonic electromagnetic field in a medium with conducting σ, permittivity ε, and permeability μ being all constants. We also assume that $\vec{J}_s$ is the time harmonic source current in a limiued region in the medium. In this case, the Maxwell's equations with time factor $e^{j\omega t}$ can be written in the time harmonic forms:

$$\nabla\times\vec{H}=\vec{J}_s+\sigma\vec{E}+j\omega\vec{D},\tag{1}$$

$$\nabla\times\vec{E}=-j\omega\vec{B},\tag{2}$$

$$\nabla\cdot\vec{B}=0,\tag{3}$$

$$\nabla\cdot\vec{D}=\rho_e.\tag{4}$$

Where $\vec{H}$ is the magnetic field intensity vector, $\vec{E}$ is

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the electric field intensity vector, $\vec{B}$ is the magnetic induction vector, $\vec{D}$ is the displacement flux vector, ρ_e is the charge density, ω represents the angular frequency of the time harmonic electromagnetic field, ∇ denotes the Hamilton operator and $j = \sqrt{-1}$.

Using Eq.(3) results in the definition of magnetic vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}. \tag{5}$$

Substituting (5) into (2), we obtain

$$\nabla \times (\vec{E} + j\omega\vec{A}) = 0 \tag{6}$$

and we can introduce an electric scalar potential φ satisfying

$$\vec{E} + j\omega\vec{A} = -\nabla\varphi . \tag{7}$$

Let
$$\vec{A}^* = \vec{A} + \frac{1}{j\omega}\nabla\varphi , \tag{8}$$

then the electric and magnetic field intensity vectors can be written as

$$\vec{E} = -j\omega\vec{A}^* \tag{9}$$

and
$$\vec{H} = \frac{1}{\mu}\nabla \times \vec{A}^* , \tag{10}$$

where $\vec{A}^*$ is known as modified magnetic vector potential. In addition, using Eq.(1) we obtain

$$\nabla \cdot (\nabla \times \vec{H})) = \mu\nabla \cdot \vec{J}_s - j\omega\mu(\sigma + j\omega\varepsilon)\nabla \cdot \vec{A}^* \tag{11}$$

and
$$\nabla \cdot \vec{A}^* = -\frac{\mu}{k^2}\nabla \cdot \vec{J}_s , \tag{12}$$

because $\nabla \cdot (\nabla \times \vec{H}) = 0$. In Eq.(12), the factor $k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)$. It may be mentioned that the current density $\vec{J}_s$ governs whether the value of $\nabla \cdot \vec{A}^*$ will be zero or not , and it is clearly unnecessary to specify any gauge of $\vec{A}^*$.

The continuity of currents leads to

$$\nabla \cdot \vec{H}_s = -\nabla \cdot (\vec{J}_e + \vec{J}_d) . \tag{13}$$

where $\vec{J}_e$ is the eddy current density and $\vec{J}_d$ is the displacement current density. It can be seen from Eq.(13) that , in general, $\nabla \cdot \vec{J}_s \neq 0$, and thus $\nabla \cdot \vec{A}^* \neq 0$. In this case, the normal component A_n^* of $\vec{A}^*$ is discontinuous across the interface between any two regions that are filled with different mediums, say $A_{1n}^* \neq A_{2n}^*$, and $\vec{W}$ satisfies $\vec{A}^* = \nabla \times \vec{W}$.

2 Boundary Value Problem of $\vec{A}^*$

Consider a three – dimensional medium – filled region V_1 with medium parameters of σ_1, ε_1, and μ_1. The medium – filled region V_1 is assumed to be bounded with medium parameters of σ_2, ε_2, and μ_2, and the source current $\vec{J}_s$ is distributed in a limited region, as shown in Fig.1. In Fig.1, S_1 represents the surface of V_1 with $\vec{n}_1$ being the outward normal unit vector of S_1. In addition, the bounded field region $\Omega = V_1 + V_2$ includes the whole field region of interest. It is worth nothing that S_2 is the surface of Ω with $\vec{n}_2$ being the outward normal unit vector of S_2. Therefore, the field boundary value problem can be written in the following form for the field region shown in Fig.1:

$$\nabla \times \vec{H} = \vec{J}_s + (\sigma + j\omega\varepsilon)\vec{E} \quad \text{in } \Omega ; \tag{14}$$

$$\nabla \times \vec{H} = -j\omega\mu\vec{H} \quad \text{in } \Omega ; \tag{15}$$

$$(\vec{E}_1 - \vec{E}_2) \times \vec{n}_1 = 0 \quad \text{on } S_1 ; \tag{16}$$

$$(\vec{E}_1 - \vec{E}_2) \times \vec{n}_1 = \vec{K} \quad \text{on } S_1 ; \tag{17}$$

$$\vec{E}_2 \times \vec{n}_2 = \vec{\alpha} \quad \text{on } S_2 ; \tag{18}$$

$$\vec{H}_2 \times \vec{n}_2 = \vec{\beta} \quad \text{on } S_2 ; \tag{19}$$

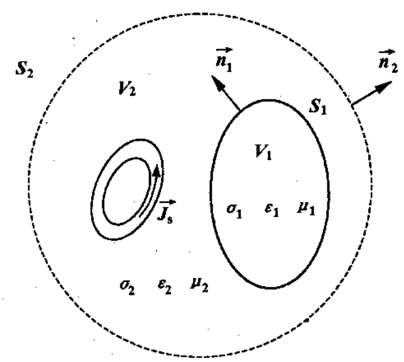

Fig.1 Typical three – dimensional time harmonic electromagnetic fields

where $\vec{K}$ is the surface conducting current density on S_1 and $\vec{\alpha}$ and $\vec{\beta}$ are given vector functions.

On the basis of Helmholtz theorem[4], to determine an unique vector field, both divergence and curl of the vector field should be specified simultaneously in the field region, and the boundary conditions should also be given. The curl $\nabla \times \vec{A}^*$ satisfies an equation that can be obtained from Eq.(14), while the divergence $\nabla \cdot \vec{A}^*$ satisfies an equation that is given by (12).

Therefore, using the boundary conditions (16), (17), (18), and (19) the boundary value problem of $\vec{A}^*$ in Fig. 1 can be written as follow:

$$\nabla \times \nabla \times \vec{A}^* - k^2 \vec{A}^* = \mu \vec{J}_s \quad \text{in } \Omega \text{ ;} \tag{20}$$

$$\nabla \cdot \vec{A}^* = -\frac{\mu}{k^2} \nabla \cdot \boldsymbol{J}_s \quad \text{in } \Omega \text{ ;} \tag{21}$$

$$(\vec{A}_1^* - \vec{A}_2^*) \times \vec{n}_1 = 0 \quad \text{on } S_1 \text{ ;} \tag{22}$$

$$\left(\frac{1}{\mu_1} \nabla \times \vec{A}_1^* - \frac{1}{\mu_2} \nabla \times \vec{A}_2^*\right) \times \vec{n}_1 = \vec{K} \quad \text{on } S_1 \text{ ;} \tag{23}$$

$$\vec{A}_2^* \times \vec{n}_2 = \vec{\alpha} \quad \text{on } S_2 \text{ ;} \tag{24}$$

$$\frac{1}{\mu_2}(\nabla \times \vec{A}_2^*) \times \vec{n}_2 = \vec{\beta} \quad \text{on } S_2 . \tag{25}$$

3 Uniqueness of the Solution of $\vec{A}^*$

Two solutions for the boundary value problem (20) ~ (25) are assumed to be $\vec{A}_a^*$ and $\vec{A}_b^*$, and the vector $\vec{F}$ is assumed to be $\vec{F} = \vec{A}_a^* - \vec{A}_b^*$. Then the vector $\vec{F}$ satisfies the following boundary value problem:

$$\nabla \times \nabla \times \vec{F} - k^2 \vec{F} = 0 \quad \text{in } \Omega \text{ ;} \tag{26}$$

$$\nabla \cdot \vec{F} = 0 \quad \text{in } \Omega \text{ ;} \tag{27}$$

$$(\vec{F}_1 - \vec{F}_2) \times \vec{n}_1 = 0 \quad \text{on } S_1 \text{ ;} \tag{28}$$

$$\left(\frac{1}{\mu_1} \nabla \times \vec{F}_1 - \frac{1}{\mu_2} \nabla \times \vec{F}_2\right) \times \vec{n}_1 = 0 \text{ on } S_1 \text{ ;} \tag{29}$$

$$\vec{F}_2 \times \vec{n}_2 = 0 \quad \text{on } S_2 \text{ ;} \tag{30}$$

$$\left(\frac{1}{\mu_2} \nabla \times \vec{F}_2\right) \times \vec{n}_2 = 0 \quad \text{on } S_2 \text{ ;} \tag{31}$$

where $\vec{F}_1 = \vec{A}_{a1}^* - \vec{A}_{b1}^*$ and $\vec{F}_2 = \vec{A}_{a2}^* - \vec{A}_{b2}^*$. In addition, letting $\vec{F}^c$ be the complex conjugate of $\vec{F}$, one can start with the following expression to analyze the problem:

$$\int_\Omega \left[\vec{F}^c \cdot \left(\nabla \times \frac{1}{\mu} \nabla \times \vec{F}\right) - \frac{1}{\mu} \nabla \times \vec{F} \cdot \nabla \times \vec{F}^c \right] \mathrm{d}\Omega = \int_{V_1} \left[\vec{F}_1^c \cdot \left(\nabla \times \frac{1}{\mu_1} \nabla \times \vec{F}_1\right) - \frac{1}{\mu_1} \nabla \times \vec{F}_1 \cdot \nabla \times \vec{F}_1^c \right] \mathrm{d}V_1 + \int_{V_2} \left[\vec{F}_2^c \cdot \left(\nabla \times \frac{1}{\mu_2} \nabla \times \vec{F}_2\right) - \frac{1}{\mu_2} \nabla \times \vec{F}_2 \cdot \nabla \times \vec{F}_2^c \right] \mathrm{d}V_2 . \tag{32}$$

Using $\vec{P} = \frac{1}{\mu} \nabla \times \vec{F}$, $\vec{Q} = \vec{F}^c$, the integral formula

$$\int_V [\vec{Q} \cdot (\nabla \times \vec{P}) - \vec{P} \cdot (\nabla \times \vec{Q})] \mathrm{d}V = \oint_s \vec{P} \cdot (\vec{Q} \times \vec{n}) \mathrm{d}S , \tag{33}$$

and the boundary condition (28), one can rewrite the right-hand side of Eq. (32):

$$\oint_{s_1} \left(\frac{1}{\mu_1} \nabla \times \vec{F}_1\right) \cdot (\vec{F}_1^c \times \vec{n}_1) \mathrm{d}S_1 - \oint_{s_1} \left(\frac{1}{\mu_2} \nabla \times \vec{F}_2\right) \cdot (\vec{F}_2^c \times \vec{n}_1) \mathrm{d}S_1 + \oint_{s_2} \left(\frac{1}{\mu_2} \nabla \times \vec{F}_2\right) \cdot (\vec{F}_2^c \times \vec{n}_2) \mathrm{d}S_2 = -\oint_{s_1} \left[\left(\frac{1}{\mu_1} \nabla \times \vec{F}_1 - \frac{1}{\mu_2} \nabla \times \vec{F}_2\right) \times \vec{n}_1 \right] \cdot \vec{F}_1^c \mathrm{d}S_1 + \frac{4\pi}{\mu_2} \lim_{r \to \infty} (r \nabla \times \vec{F}_2^c)(r \vec{F}_2^c \times \vec{n}_2) , \tag{34}$$

Substituting boundary conditions (29), (31), and (32) into Eq. (34), one can find that the right-hand side of Eq. (32) equals zero.

On account of Eq. (26), the left-hand side of Eq. (32) can be written as

$$\int_\Omega \left[\vec{F}^c \cdot \left(\nabla \times \frac{1}{\mu} \nabla \times \vec{F}\right) - \frac{1}{\mu} \nabla \times \vec{F} \cdot \nabla \times \vec{F}^c \right] \mathrm{d}\Omega = \int_\Omega \frac{1}{\mu} (k^2 \vec{F}^c \cdot \vec{F} - \nabla \times \vec{F} \cdot \nabla \times \vec{F}^c) \mathrm{d}\Omega = \int_\Omega \left(\omega^2 \varepsilon \mid \vec{F} \mid^2 - \frac{1}{\mu} \mid \nabla \times \vec{F} \mid^2 - \mathrm{j}\omega\sigma \mid \vec{F} \mid^2\right) \mathrm{d}\Omega . \tag{35}$$

Because the right-hand side of Eq. (32) equals zero, the real and imaginary parts of Eq. (35) are

$$\int_\Omega \left(\omega^2 \varepsilon \mid \vec{F} \mid^2 - \frac{1}{\mu} \mid \nabla \times \vec{F} \mid^2\right) \mathrm{d}\Omega = 0 \tag{36}$$

and

$$\int_\Omega \omega\sigma \mid \vec{F} \mid^2 \mathrm{d}\Omega = 0 . \tag{37}$$

respectively, here $\mu > 0$, $\omega > 0$, $|\vec{F}| \geqslant 0$ and $|\nabla \times \vec{F}| \geqslant 0$.

We can see from (37) that $|\vec{F}| = 0$ when $\sigma > 0$, and using $\vec{F} = 0$ in (36) result in $|\nabla \times \vec{F}| = 0$, therefore $\vec{F} = 0$ for $\sigma > 0$. Particularly, for $\sigma = 0$, we can treat this case as a limit that $\sigma > 0$ and σ approaches zero, thus $|\vec{F}| = 0$. These statements imply that the boundary value problem (20) ~ (25) has a unique solution.

4 Verifying Example

To verify the divergence expression (12) and the boundary value problem (20) ~ (25), we solve the Sommerfeld's half-space problem to give analytical solution of $\vec{A}^*$, as shown in Fig.2.

In Fig.2, (ρ, θ, z) are cylindrical coordinates, and $\vec{e}_\rho, \vec{e}_\theta$ and $\vec{e}_z$ are the usual unit vectors of the cylindrical coordinate system. The region 1 implies $z > 0$ with medium parameters $\sigma_1, \varepsilon_1, \mu_0$; the region 2 implies $z < 0$ with medium parameters $\sigma_2, \varepsilon_2, \mu_0$. At $\vec{r}_0 = z_0\vec{e}_z$ of region 2, one can find a vertical time harmonic current element $I\ l$, whose current density can be expressed in terms of δ function:

$$\vec{J}_s(\vec{r}) = I\ l\delta(|\vec{r} - \vec{r}_0|)\vec{e}_z, \quad (38)$$

Where $\vec{r}$ is a vector from original point O to the point of interest.

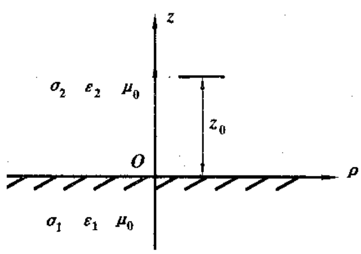

Fig.2 The vertical current element in a homogeneous half - space conducting medium

The equations of field region shown in Fig.2 can be written in the following forms:

$$\nabla \times \nabla \times \vec{A}_1^* - k_1^2\vec{A}_1^* = 0 \quad (z < 0); \quad (39)$$

$$\nabla \cdot \vec{A}_1^* = 0 \quad (z < 0); \quad (40)$$

$$\nabla \times \nabla \times \vec{A}_2^* - k_2^4\vec{A}_2^* = \mu_0 I\ l\delta(|\vec{r} - \vec{r}_0|) \quad (z > 0); \quad (41)$$

$$\nabla \cdot \vec{A}_2^* = -\frac{\mu_0 I\ l}{k_2^2}\frac{\partial}{\partial z}\delta(|\vec{r} - \vec{r}_0|) \quad (z > 0); \quad (42)$$

The boundary conditions are

$$A_{1\rho}^* = A_{2\rho}^* \quad (z = 0), \quad (43)$$

$$\frac{\partial}{\partial z}A_{1\rho}^* - \frac{\partial}{\partial \rho}A_{1z}^* = \frac{\partial}{\partial z}A_{2\rho}^* - \frac{\partial}{\partial \rho}A_{2z}^* (z = 0). \quad (44)$$

It is noticed that the values of fields at the infinite points are finite. We can obtain the following analytical solution of this problem using a derivation(given in the Appendix):

$$\vec{A}_1^* = h_1\int_0^\infty \lambda u_1 f\left[-\vec{e}_\rho J_1(\lambda\rho) + \vec{e}_z\frac{\lambda}{u_1}J_0(\lambda\rho)\right] e^{u_1 z}d\lambda, \quad (45)$$

$$\vec{A}_2^* = \left[\vec{e}_\rho\frac{\partial^2}{\partial\rho\,\partial z} + \vec{e}_z\left(k_2^2 + \frac{\partial^2}{\partial z^2}\right)\right]\frac{e^{-jk_2R}}{R} + h_2\int_0^\infty \lambda^2 g\left[\vec{e}_\rho J_1(\lambda\rho) + \vec{e}_z\frac{\lambda}{u_2}J_0(\lambda\rho)\right] e^{-u_2 z}d\lambda, \quad (46)$$

where

$$h_i = \frac{\mu_0 I\ l}{4\pi k_1^2} \quad (i = 1,2); \quad (47)$$

$$k_i^2 = -j\omega\mu_0(\sigma_i + j\omega\varepsilon_i) \quad (i = 1,2); \quad (48)$$

$$u_i = (\lambda^2 - k_i^2)^{1/2} \quad (i = 1,2); \quad (49)$$

$$R = [\rho^2 + (z - z_0)^2]^{1/2}; \quad (50)$$

$$N = k_1^2 u_2 + k_2^2 u_1; \quad (51)$$

$$f = \frac{2\lambda k_1^2}{N}e^{-u_2 z_0}; \quad (52)$$

$$g = \frac{k_1^2 u_2 - k_2^2 u_1}{N}e^{-u_2 z_0}. \quad (53)$$

Also, J_0 and J_1 are the Bessel functions of order 0 and order 1 respectively.

The radiation problem of vertical time harmonic current element shown in Fig.2 has been deeply investigated in history. Reference [5] states the Hertz vector $\vec{\prod}$ which could be used to solve the boundary value problems. Using the method provided by Ref. [5], one can obtain the solution of $\vec{\prod}$ in Fig.2:

$$\vec{\prod}_1 = 2w_1 k_1^2\int_0^\infty \frac{\lambda}{N}J_0(\lambda\rho)e^{u_1 z - u_2 z_0}d\lambda\,\vec{e}_z \quad (z < 0); \quad (54)$$

$$\vec{\prod}_2 = w_2\left[\frac{e^{-jk_2R}}{R} + \frac{e^{-jk_2R_s}}{R_s} - 2k_2^2\int_0^\infty \frac{\lambda}{N}\cdot\frac{u_1}{u_2}J_0(\lambda\rho)e^{-u_2(z+z_0)}d\lambda\right]\vec{e}_z \quad (z > 0), \quad (55)$$

Where

$$w_i = j\omega h_i \quad (i = 1,2), \quad (56)$$

$$R_s = [\rho^2 + (z + z_0)^2]^{1/2}. \quad (57)$$

It is easy to verify that substituting (54) and (55) into

$$\vec{A}^* = -\frac{1}{j\omega}(k^2 + \nabla\nabla\cdot)\vec{\prod}, \quad (58)$$

respectively, one can obtain the same expressions of (45) and (46), which implie that (12) and boundary value problem (20) ~ (25) are valid and reliable.

5 Conclusions

The divergence expression of modified magnetic vector potential $\vec{A}^*$ is determined by the basic law of time harmonic electromagnetic fields, therefore, it is not necessary to specify any gauge of $\vec{A}^*$. Generally speaking, $\nabla \cdot \vec{A}^* \neq 0$.

This paper gives the boundary value problem of modified magnetic vector potential $\vec{A}^*$, which can be used to solve the three – dimensional time harmonic electromagnetic fields in any arbitrarily complex regions including subregions filled with linear homogeneous mediums.

Appendix: Analytical Solution to the Sommerfeld's Half – space Problem

In the field region of Fig. 2, $\vec{A}^*$ can be expressed as

$$\vec{A}^* = \vec{A}_\rho^* \vec{e}_\rho + \vec{A}_z^* \vec{e}_z \text{ and } \frac{\partial}{\partial \theta}\vec{A}^* = 0.$$

A. The General Solutions of Equations (39) and (41)

Because

$$\nabla \times \nabla \times \vec{A}^* = \nabla(\nabla \cdot \vec{A}^*) - \nabla^2 \vec{A}^*, \tag{A1}$$

the Eq. (39) and (41) can be written as

$$(\nabla^2 + k_1^2)\vec{A}_1^* = 0 \tag{A2}$$

and

$$(\nabla^2 + k_2^2)\vec{A}_2^* = -\mu_0 Il\left(\vec{e}_z + \frac{1}{k_2^2}\nabla\frac{\partial}{\partial z}\right)\delta(|\vec{r} - \vec{r}_0|), \tag{A3}$$

respectively, We assume that the vector $\bar{S}$ indicates the right – hand side of Eq. (A3). Using the three – dimensional free – space Green's function – satisfied identical equation

$$\delta(|\vec{r} - \vec{r}'|) = \frac{1}{4\pi}(\nabla^2 + k^2)\frac{e^{-jk|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \tag{A4}$$

and

$$\frac{\partial}{\partial \rho}\nabla^2 = \left(\nabla^2 - \frac{1}{\rho^2}\right)\frac{\partial}{\partial \rho}, \tag{A5}$$

one can obtain

$$\vec{S} = h_2\left[\vec{e}_\rho\left(\nabla^2 - \frac{1}{\rho^2} + k_2^2\right)\frac{\partial^2}{\partial\rho\partial z} + \vec{e}_z(\nabla^2 + k_2^2)\left(k_2^2 + \frac{\partial^2}{\partial z^2}\right)\right]\frac{e^{-jk_2R}}{R}. \tag{A6}$$

Therefore, the equation of components, $A_{1\rho}^*$, A_{1z}^*, $A_{2\rho}^*$, and A_{2z}^*, can be written as

$$\left(\nabla^2 - \frac{1}{\rho^2} + k_1^2\right)A_{1\rho}^* = 0, \tag{A7}$$

$$(\nabla^2 + k_1^2)A_{1z}^* = 0, \tag{A8}$$

$$\left((\nabla^2 - \frac{1}{\rho^2 + k_2^2}\right)\left(A_{2\rho}^* - h_2\frac{\partial^2}{\partial\rho\partial z}\frac{e^{-jk_2R}}{R}\right) = 0 \tag{A9}$$

and

$$(\nabla^2 + k_2^2)\left[A_{2z}^* - h_2\left(k_2^2 + \frac{\partial^2}{\partial z^2}\right)\frac{e^{-jk_2R}}{R}\right] = 0. \tag{A10}$$

The general solutions of (A7), (A8), (A9), and (A10) can be obtained by using the separation of variables[1]:

$$A_{1\rho}^* = \int_0^\infty J_1(\lambda\rho)(a_1 e^{-u_1 z} + a_2 e^{u_1 z})d\lambda \ (z < 0), \tag{A11}$$

$$A^{1z} = \int_0^\infty J_0(\lambda\rho)(b_1 e^{-u_1 z} + b_2 e^{u_1 z})d\lambda \quad (z < 0), \tag{A12}$$

$$A_{2\rho}^* = h_2\frac{\partial^2}{\partial\rho\partial z}\frac{e^{-jk_2R}}{R} + \int_0^\infty J_1(\lambda\rho)(c_1 e^{-u_2 z} + c_2 e^{u_2 z})d\lambda \quad (z > 0), \tag{A13}$$

$$A_{2z}^* = h_2\left(k_2^2 + \frac{\partial^2}{\partial z^2}\right)\frac{e^{-jk_2R}}{R} + \int_0^\infty J_0(\lambda\rho)(d_1 e^{-u_2 z} + d_2 e^{u_2 z})d\lambda \quad (z > 0), \tag{A14}$$

where a_i, b_i, c_i and d_i are the undetermined constants with $i = 1, 2, 3, 4$; J_n is the Bessel function of order n of the first kind,

B. The Determination of Constants a_i, b_i, c_i and d_i

We assume that the real parts of u_i and k_i are positive. Because the values of fields are finite at the infinite points, we can obtain

$$a_1 = 0,\ b_1 = 0;$$
$$c_2 = 0,\ d_2 = 0,$$

and only four constants, a_2, b_2, c_1 and d_1, need to be determined.

Substituting (A11) and (A12) into (40) results in

$$\lambda a_2 + u_1 b_2 = 0 \tag{A15}$$

Letting M be the right – hand side of (42), and using (A4), one can get

$$M = h_2\frac{\partial}{\partial z}(\nabla^2 + k_2^2)\frac{e^{-jk_2R}}{R}. \tag{A16}$$

In addition, substituting (A13), (A14) and (A16) into (42), one can write

$$\lambda c_1 - u_2 d_1 = 0. \tag{A17}$$

Employing the Sommerfeld's integral formula[5]

$$\frac{e^{-jk\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} = \int_0^\infty \frac{\lambda}{\sqrt{\lambda^2-k^2}} J_0(\lambda\rho) e^{-\sqrt{\lambda^2-k^2}|z|} d\lambda, \quad \text{(A18)}$$

all terms of right - hand side of (A13) can be expressed in the integral forms, and in this case the boundary condition (43) leads to

$$a_2 - c_1 = -h_2\lambda^2 e^{-u_2 z_0}. \quad \text{(A19)}$$

Similarly, the right - hand side of (A14) can also be expressed in the integral form by using (A18), and the boundary condition (44) leads to

$$u_1 a_2 + \lambda b_2 + u_2 c_1 - \lambda d_1 = \frac{h_2 k_2^2 \lambda^2}{u_2} e^{-u_2 z_0}. \quad \text{(A20)}$$

Therefore, using (A15), (A17), (A19) and (A20), one can obtain

$$a_2 = -h_1\lambda u_1 f, \; b_2 = h_1\lambda^2 f,$$

$$c_1 = h_2\lambda^2 g, \; d_1 = \frac{h_2\lambda^3}{u_2} g .$$

Substituting the expressions of a_2, b_2, c_1 and d_1 into (A11), (A12), (A13) and (A14), one will get the results of (45) and (46).

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修正矢量磁位的散度及其边值问题

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摘　要:根据时谐形式的麦克斯韦方程组,导出了均匀媒质中修正矢量磁位 $\vec{A}^*$ 的散度表达式.在此基础上,建立了 $\vec{A}^*$ 的边值问题,并证明 $\vec{A}^*$ 解答的唯一性,最后以有解析解的时谐场为例验证了本文建立的边值问题的正确性.

关键词:修正矢量磁位;规范;边值问题;时谐电磁场